Quantum Graphity

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We introduce a new model of background independent physics in which the degrees of freedom live on a complete graph and the physics is invariant under the permutations of all the points. We argue that the model has a low energy phase in which physics on a low dimensional lattice emerges and the permutation symmetry is broken to the translation group of that lattice. In the high temperature, or disordered, phase the permutation symmetry is respected and the average distance between degrees of freedom is small. This may serve as a tractable model for the emergence of classical geometry in background independent models of spacetime. We use this model to argue for a cosmological scenario in which the universe underwent a transition from the high to the low temperature phase, thus avoiding the horizon problem.

Contents

I. INTRODUCTION

Background independent theories of quantum gravity are those that share with general relativity the property that the formulation of the laws of the theory does not require the specification of any classical metric geometry or boundary conditions. As a result, such theories are often formulated in the language of combinatorics and representation theory, as that is what is left of quantum mechanics when reference to manifolds and global symmetries is removed.

The biggest problem such approaches face is to demonstrate that classical general relativity emerges from them at low energies or large volumes. There is some evidence for this, in several approaches including causal dynamical triangulations [1], spin foam models [2] and loop quantum gravity [3], but the problem is certainly not yet definitively solved.

One purpose of this paper is to suggest that it may help to take a more physical approach to this problem. This begins by positing that interesting models of quantum spacetime will have at least two thermodynamic phases. In the high temperature, or disordered phase, notions of geometry and perhaps even dimension and topology are useless and the physics must be described in purely quantum mechanical terms. In the low temperature phase, the system becomes ordered in such a way that it can be described in terms of fields living on a low dimensional spacetime manifold with metric obeying Einstein's equations, to a suitable approximation.

Seeing it this way may be helpful, because it may allow us to attack the problem of the emergence of spacetime in the low temperature phase with tools from statistical physics. But there is another reason for interest in such a scenario, which is that we know that the universe has cooled from an initially very hot state. It is then possible that the universe at some early time was in the high temperature phase and underwent a transition to the low temperature phase at some time t_c. Since geometry is an emergent property of the low temperature phase we can call the transition *"geometrogenesis."* This event may have set up conditions which are observable now in detailed observations of the cosmic microwave background and large scale structure. If so it may be possible that such a scenario provides an alternative to inflation as an explanation of the cosmological observations.

For example, in loop quantum gravity, the states are described by graphs. Typical graphs, in this theory, as well as generally, have high interconnectivity and do not admit of an easy description in terms of a discrete geometry, nor do they easily embed in, or coarse grain to, low dimensional geometries. They have small diameters (the maximal distance between nodes, counted by graph links separating them.) It is then natural to assume that the high temperature phase is dominated by such non-geometrical highly connected graphs. We may note that this may play the role of an inflationary phase, in ensuring that when the classical spacetime emerges, all regions of space arise from parts of the graph that were in causal contact in the earlier phase.

The physical question is then why graphs of low connectivity, low valence and large diameter should dominate in the low temperature phase. A second question is whether this scenario has consequences for cosmological observations.

Still another set of questions has to do with the role of emergent symmetry in the phase transition to the ordered phase. In [7] it was shown that many models of dynamical quantum geometry have emergent degrees of freedom which constitute noiseless subsystems. These exist due to emergent symmetries which become apparent only when the quantum system is analyzed by dividing it into subsystems and environment. These emergent particles carry conserved quantum numbers and hence, as described in [8], are candidates for elementary particles. One proposal for how space emerges is then that it is defined by the interactions of these emergent particles. Using this language, we can then anticipate that the transition to the low temperature phase will be accompanied by an expansion in the Hilbert space dimensions of these noiseless subsystems, corresponding to the emergence of translational and rotational invariances.

Still another question raised by the scenario just discussed is whether the transition to an ordered phase characterized by the emergence of local geometrical structure must be complete. Is it possible that after the transition there will remain defects in or disorderings of locality[7]? What this means is that the state after the phase transition may be dominated by graphs which only approximately embed in low dimensional geometries. Defects in locality would arise when two nodes of a low dimensional graph, which are far away in the approximate classical metric, are connected. One way to say this is that the notion of locality encoded in the graph may not completely coincide with the notion of locality given by the emergent metric that describes its course grained properties.

In [14] the implications of this possibility for cosmology are investigated and it is found that there may indeed be striking observational consequences of disordeded locality. This leads us to ask another question about the phase transition from which space emerges: is it possible that the result is a universe with disordered locality?

To investigate all these questions, we decided to invent a model which captures the key features of the scenario we have just described, while being easier to analyze than full quantum gravity models such as spin foam models. The purpose of this paper is to describe such a model and begin the investigation of its properties.

The model described here, which we call *"quantum graphity,"* is based on the complete graph on N nodes. This means that every two nodes in our graph are connected by an edge. The degrees of freedom live on the edges of the graph and the dynamics is invariant under the group of permutations of the N nodes. There is a ground state for each edge which signifies that the edge is turned off, and excited states which indicate that the edge is on and in various states. In the model we discuss here, there are three "on" states for each edge corresponding to the states of a spin-one system. Thus, the states of the system include every graph on N nodes.

We choose the Hamiltonian so that the ground state of the model breaks the permutation symmetry by the formation of a low dimensional lattice. We argue (but do not prove) that under certain conditions the spins in the system can arrange themselves in regular, lattice-like patterns at low temperatures. When the graph is frozen, the model is closely related to a model of Levin and Wen [4, 5, 6] which has emergent gauge degrees of freedom. The excitations of the spin system are interpreted as photons coupled to massive charged particles and propagating on the graph consisting of the "on" edges of the graph.

The outline of the paper is as follows. In section II, we introduce a classical and a quantum model and explain the various terms and constants in the proposed Hamiltonian. We discuss some properties of the models in the high and low temperature regimes in section III, and discuss the emergence of photons in section IV. The implications of the model for cosmology are discussed in section V. We summarize in section VI.

II. THE MODEL

In this section, we introduce the "quantum graphity" model based on the complete graph of N nodes. We begin by describing a classical model and then extend it to a quantum mechanical setting.

A. Classical model

A complete graph on N nodes is a collection of N points, labeled $a, b, \ldots$, which are all connected to each other by edges. We make the edges carry labels J and M in the following possible configurations

$$(J,\, M) \in \{(0,\, 0),\, (1,\, -1),\, (1,\, 0),\, (1,\, 1)\}. \tag{1}$$

We interpret the label $(0,\, 0)$ to signify there is no link between two points, and the remaining labels to signify there is a link. We then call the state $(0,\ 0)$ to be an "off" state, while the three remaining states are "on" states.

We consider a Hamiltonian

$$H = H_{links} + H_{vertices} + H_{loops} + H_{hop} + H_{LQG}. \tag{2}$$

Let us explain the terms of H, in order. The first term is

$$H_{links} = V \sum_a \left(v_0 - \sum_b J_{ab} \right)^2 ; \tag{3}$$

where V is a positive coupling constant, v_0 is a fixed number, and the sums are over all points in the graph. The minimum of H_{links} occurs when the number of "on" links adjacent to every node in the graph is v_0. Thus this term term tells us that the ground state will consist of graphs with valence v_0.

The second term is

$$H_{vertices} = C \sum_a \left(\sum_b M_{ab} \right)^2 + D \sum_{ab} M_{ab}^2 \tag{4}$$

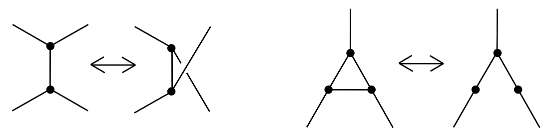
In the first line, the C term favors configurations in which the m values of spins at each vertex add up to zero. The D term gives preference to configurations in which all the said spins have $m = 0$.

The third term in the Hamiltonian is

$$H_{loops} = - \sum_{\text{minimal loops}} \frac{1}{L!}\, B(L) \prod_{i=1}^{L} M_i. \tag{5}$$

Here, the sum is over minimal loops. We define these to be loops that cannot be factored into the product of two loops of "on" edges that contain some of the same edges. The products are defined over a closed sequence of edges as follows

$$\prod_{i=1}^{L} M_i \;=\; M_{ab}\, M_{bc}\, \ldots M_{yz}\, M_{za}. \tag{6}$$



(a) (b)

FIG. 1: Examples of terms in Hamiltonian H_{LQG} that acts on j variables. (a) Exchange of neighboring links. (b) Addition or subtraction of an edge.

L is the length of the loop, and we note that a given on graph defined by an assignment of J's can have mimimal loops of varying lengths.

The coupling $B(L)$ is assumed to take the form

$$B(L) = B_0 B^L \tag{7}$$

where B_0 is a positive coupling constant and B is dimensionless. The separation of B from B_0 is useful because B can be now associated with each instance of M in the loop product (6). We note then that the overall coefficient of a loop term is proportional to $B^L/L!$. It is thus small at very low and at very high L, but has a maximum value at some particular L_*,

$$\frac{B^{L_*}}{L_*!} > \frac{B^{L'}}{L'!} \qquad \forall L' \neq L_*. \tag{8}$$

We call L_* the preferred loop length.

In comparison with H_{links}, note that H_{loops} has an overall minus sign. Note also that H_{loops} contributes nothing to the energy unless the edges it acts upon are in one of the "on" states. Thus there is a competition between this term and H_{links} which will be responsible for fixing the assignment of "on" edges.

The last two terms in the Hamiltonian are H_{hop} and H_{LQG}. We will not need to specify these in detail, except to say that H_{hop} allows the M variables to propagate from an on edge to other edges adjacent to the same vertex, and the term H_{LQG} generates local moves that turn edges on and off and thus allow one configuration of "on" links to morph into another one. The action of these graph-changing terms is illustrated in Figure 1. We assume that the couplings characterizing these terms are such that they do not significantly alter the equilibrium and ground states of the model.

B. Quantum model

We now introduce a quantum model similar to the one we have just described by turning the configuration space into a Hilbert space. On each edge we put a four state Hilbert space, $\mathcal{H}_{spin}$, which is spanned by an orthonormal basis of states $|j, m\rangle$,

$$\mathcal{H}_{spin} = \mathrm{span}\,\{|0,0\rangle,\, |1,-1\rangle,\, |1,0\rangle,\, |1,+1\rangle\}. \tag{9}$$

Since there are $N(N-1)/2$ links in a complete graph with N points, the Hilbert space for the whole system is,

$$\mathcal{H}_{tot} = \otimes^{N(N-1)/2}\, \mathcal{H}_{spin}. \tag{10}$$

As in the classical model, we will interpret states with $j_{ab} = 0$ ($j_{ab} = 1$) as indicating the absence (presence) of a link between points a and b.

We define four operators acting on the Hilbert space of each spin[1]. The first two, J and M, are eigen-operators of the states $|j,\, m\rangle$ such that

$$\begin{aligned} J\,|j,\, m\rangle &= j\,|j,\, m\rangle \\ M\,|j,\, m\rangle &= m\,|j,\, m\rangle. \end{aligned} \tag{11}$$

The other two are the lowering operator M^- and the raising operator M^+ which act as

$$\begin{aligned} \sqrt{2}\, M^+\,|j,\, m\rangle &= \sqrt{(j-m)(j+m+1)}\,|j,\, m+1\rangle \\ \sqrt{2}\, M^-\,|j,\, m\rangle &= \sqrt{(j+m)(j-m+1)}\,|j,\, m-1\rangle. \end{aligned} \tag{12}$$

In terms of algebra, the operator J commutes with M and $M^{\pm}$, which form a closed algebra among themselves

$$[M^+,\, M^-] = M, \qquad [M,\, M^{\pm}] = \pm M^{\pm}. \tag{13}$$

It will be important later that all these operators annihilate the $|0,\, 0\rangle$ state,

$$J\,|0,\, 0\rangle = M\,|0,\, 0\rangle = M^{\pm}\,|0,\, 0\rangle = 0, \tag{14}$$

and that all non-zero matrix elements have unit magnitude.

We now write the quantum Hamiltonian

$$\hat{H} = \hat{H}_{links} + \hat{H}_{vertices} + \hat{H}_{loops} + \hat{H}_{hop} + \hat{H}_{LQG}. \tag{15}$$

The first terms $\hat{H}_{links} + \hat{H}_{vertices}$ are gotten from the classical (3) and (4) by a replacement of classical values with quantum operators.

The next term, the quantum loop Hamiltonian, $\hat{H}_{loops}$, is

$$\hat{H}_{loops} = -\sum_{\text{loops}} \frac{1}{L!}\, B(L) \prod_{i=1}^{L} M_i^{\pm} \tag{16}$$

now involves sums over loops of varying even lengths, L. It is slightly different than its classical counterpart in that the product of $M_i^{\pm}$ is understood as

$$\prod_{i=1}^{L} M_i^{\pm} \;=\; M_{ab}^{+} M_{bc}^{-} \ldots M_{yz}^{+} M_{za}^{-}. \tag{17}$$

As before, the products of operators $M^{\pm}$ act on successive links along minimal links of a loop. Since the series (17) starts with M^+ and ends with M^-, the length L of the loop $a, b, \ldots, z$ must now be even, $L = 4,\, 6,\, 8,\, \ldots$. We may simplify the model by restricting

[1] The operators we define are related to the angular momentum operators J^2, J_z, and $J^{\pm}$ in their usual defintions. The two sets of operators differ however in their normalization by factors of $\sqrt{2}$, which in the present setup is equal to $\sqrt{j_{max}(j_{max}+1)}$.

to low L, for example, $L = 4, 6$. We chose the coupling $B(L)$ to be of the same form as in (7) and define the preferred loop length L_* analogously to (8).

We do not specify the details of the final two terms in $\hat{H}$. Their role is again to move M values between neighboring edges ($\hat{H}_{hop}$) and allow the graph to morph from one configuration to another ($\hat{H}_{LQG}$) by moves shown in Figure 1.

This quantum model is significantly more complex than the classical one in the previous section. One of the complications is that the loop Hamiltonian $\hat{H}_{loops}$ does not commute with $\hat{H}_{vertices}$, which implies that eigenstates of the Hamiltonian will generally be superpositions of states involving different m configurations. To understand the role of the terms $\hat{H}_{vertices}$ and $\hat{H}_{loops}$, it is helpful to first consider the graph of "on" links to be frozen in a particular configuration, say a regular cubic lattice where the minimal loops in the graph are plaquettes. In this case, these terms reduce to the rotor model of Levin and Wen [6]. We briefly describe the expected physics as this reduction will become important in section IV where we will discuss the m degrees of freedom as giving rise to a gauge theory on a lattice.

In the absence of the loop term, the ground state of $\hat{H}_{vertices}$ consists of all links having $m = 0$. Excited states appear as open or closed chains of alternating $m = +1$ and $m = -1$ links. These excitations are called strings. Their energy above the ground state is proportional to D times the number of edges forming the string. Thus the coupling D can be thought of as a string tension. Nodes on which the C term is not minimized are said to carry the ends of open strings. The energy of a string end is proportional to C. We take $C \gg D$ such that open ends occur very infrequently or not at all.

Given a graph with all "on" edges labelled by $m = 0$, a loop operator acts as to create a closed string of alternating $m = +1$ and $m = -1$ links. The string will acquire tension through the D term. However, since the sign of the B_0 term is negative, the overall energy of the state may increase or decrease and this creates the possibility of two distinct scenarios. In one scenario, the tension in a string is greater than the contribution from the loops term, so the overall effect of creating a string is to increase the energy of the system. If this is the case, then the string represents an excited state over the vacuum in which all m values set to zero. The second scenario is the one that we will be mostly interested in. There, the tension is small compared to the contribution from $\hat{H}_{loops}$ so that creating a string decreases the energy. This indicates that the true ground state of the model consists of a superposition of a large number of strings, a string condensate. We should note that because the graph has a finite number of nodes and the m values on each edge only take three possible values, the Hamiltonian is bounded from below. Hence the string-condensed ground state exists even though it is difficult to write down.

The quantum model described in this section is not defined on a regular and fixed lattice - all the terms in the Hamiltonian are invariant under permutations. Nevertheless, we can expect the same kind of competition between the D term and the B_0 term to possibly lead to string condensation.

To summarize, our proposed quantum model contains four dimensionful coupling constants (V, C, D, and B_0) and three dimensionless numbers (N, B and v_0). These parameters are presumed to satisfy some rather generic conditions such as $N \gg 1$, $C \gg D$, and $V \gg 1$. We discuss these conditions in more detail in the next section.

III. THERMODYNAMIC PHASES

We now describe the states of the graph when it is coupled to a heat bath with temperature $kT = 1/\beta$. As the model contains several coupling constants, the complete phase diagram of the model is expected to be very complex. We focus only on two extreme regimes - the very high and the very low temperature regimes - and, although we do not study their nature, suppose there to be one or more phase transitions that occur between them. We will assume for simplicity also that $V >> B, C, D$ so that the dominant term at high temperature comes from H_{links}.

A. High Temperature

When the temperature is high, $T > V$, the spins are in a disordered state and the average valence of a typical member of the thermal ensemble can be high. In this high temperature regime the system can be thought of as non-local as thermal fluctuations are capable to turning on or off links between any two points.

When $T \gg V$, we can ignore the terms in the Hamiltonian that depend on M. Since the spins fluctuate, we can also approximate the partition function Z_N for the whole system by H_{link} alone, which means that we may take

$$Z_N \approx Z_1^N, \tag{18}$$

where Z_1 is the parition function for one node only. We write Z_1 as

$$Z_1 = \sum_v 3^v \exp\left(-\beta V(v - v_0)^2\right) \tag{19}$$

with the sum over all possible valences. Since each "on" spin can take three possible m values, there is a multiplicity factor 3^v in front of the Boltzman factor. We further write

$$Z_1 = \sum_v \exp\left(-\beta f(v)\right) \tag{20}$$

as a definition of $f(v)$. It follows that

$$f(v) = V(v - v_0)^2 - vkT \ln 3 \tag{21}$$

and that its minimum occurs when

$$v = v_0 + \frac{kT}{2V} \ln 3. \tag{22}$$

Thus we find that the effective valence of a point increases linearly with temperature.

A typical graph in the ensemble then may be considered to be a random graph, characterized by a probability p that each of the $N(N-1)/2$ of the edges are excited. Given (22) and the fact that there are $N-1$ edges adjacent to each node, this probability is thus

$$p = \frac{v}{(N-1)} \sim \frac{T}{NV}. \tag{23}$$

In random graph theory there is a probability $p_0 \sim 1/N$ that marks the transition above which a typical graph is connected. The ensemble will be in this regime so long as $T > T_{connected} \sim V$. When the temperature T is on the order of NV, almost all edges in the complete graph will be "on."

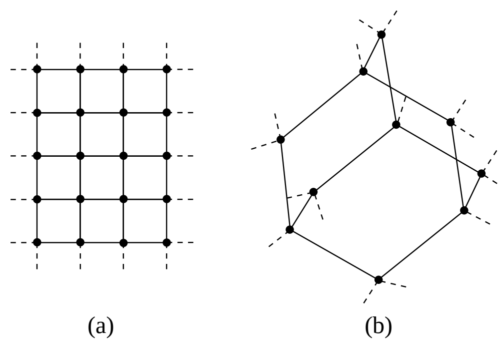


FIG. 2: Sample lattice formations with 4-valent vertices. Solid lines represent "on" links and dashed lines designate how the fragments shown fit inside a larger lattice. "Off" links are not shown.

B. Low Temperature

When the temperature is very low, $T \ll V$, the graph will essentially consist of nodes with a fixed valence v_0. Once the links Hamiltonian is at its minimum, the other terms $\hat{H}_{vertices}$ and $\hat{H}_{loops}$ gain importance. We now proceed to discuss the effect of those M sensitive terms in the Hamiltonian on the distribution of "on" links. At the same time, we assume that the amplitudes of the LQG terms are very small so that any low lying excitations do not change the patterns of "on" and "off" links. Instead the low lying excitations involve propagation of the m degrees of freedom. Hence for the rest of this section, we use the terms graph and network to refer to configurations of "on" links. We focus on the case $v_0 = 4$ for concreteness.

There are many distinct types of structures that can be built out of a 4-valent network of edges. There can be tree-like structures in which there are no closed loops, regular structures made out of polygons with four, six, or more sides, or regular and irregular structures made up of many different kinds of polygons. Graphs can be planar or non-planar. Two of these graphs, the two-dimensional square lattice and the three-dimensional diamond lattice, are shown in Figure 2 but other arrangements are possible as well. Based on valence alone, all these structures are in principle candidates for the ground state of the system at low temperatures.

The terms in $\hat{H}_{vertices}$ and $\hat{H}_{loops}$ can differentiate between these candidates. In particular, loop terms can lower the energy when the graph contains closed paths. In the classical model, this can happen when the couplings C and D are negligible and the m values on the edges are set to $+1$ or -1, or combinations of these. For a particular value of B, there will be a preferred loop length L_* and the ground state will be such that the number of loops of this length is maximized. Given a limited valence of the nodes, this maximization criterion may have the effect of creating a low-dimensional lattice. For example, if we consider only even-length loop terms and chose B such that $L_* = 4$, the low dimensional graph may resemble the square lattice shown in Figure 2(a). If we chose B such that $L_* = 6$, the diamond shaped lattice in Figure 2(b) would be preferred.

In the quantum model, the loop terms generate strings of alternating $m = +1$ and $m = -1$ edges which contribute a positive energy through the tension D term. For loops to be encouraged in the low temperature limit, we need to have the loop term be dominant over the tension. We can roughly work out when string condensation can happen given a particular graph. As a first example, we take the square grid shown in Figure 2. To consider this edge configuration as a plausible configuration, we need to take B such that $L_* = 4$. On this graph there are two edges per plaquette. Thus string condensation could occur only if the couplings satisfy the inequality

$$2D - \frac{1}{4!} B_0 B^4 < 0. \tag{24}$$

Alternatively, we can introduce a parameter $\gamma > 1$ and set

$$48\gamma D = B_0 B^4. \tag{25}$$

When these requirements are satisfied, we can argue that the ground state lattice should be square and that it should support string condensation. As a different example, consider the diamond lattice shown in Figure 2(b). Diamond is characterized by hexagonal plaquettes, so we chose B such that $L_* = 6$. The requirement for string condensation should thus be

$$2D - \frac{1}{6!} B_0 B^6 < 0 \tag{26}$$

because there are again two edges per plaquette.

It is reasonable to ask whether the lattices considered in Figure 2 are truly the ground states of the model for the parameter ranges specified. We cannot fully answer this question at this point but we can offer additional observations which support our proposal. If we think of the ground state lattice to crystallize by evolving via the moves of Figure 1 and note that these moves do not disconnect graphs and act equally on all possible edge configurations, then we would be led to hypothesize that the ground state lattice should be connected and homogenous. The configurations in Figure 2 thus seem to be good candidates. The diamond lattice seems like a particularly good candidate as each pair of edges at a vertex supports a plaquette.

We emphasize that, in the quantum model, the formation of a lattice at low temperatures must be accompanied by string condensation. We discuss what this means in more detail in the next section.

IV. THE EMERGENCE OF GAUGE FIELDS AT LOW TEMPERATURE

Once the link degrees of freedom freeze, we are left just with m degrees of freedom. It turns out these give a lattice gauge theory [4, 5, 6].

We consider a phase in which the links are arranged in a regular three-dimensional pattern of four-sided plaquettes and there are no open ended strings. This could happen for example when the valence of a node is set to $v_0 = 6$. In this phase, the V and C terms in the Hamiltonian are constant and we ignore them[2]. The non-vanishing terms then consist of the tension and loop operators with exactly four edges,

$$H_{lowT} \sim D \sum_{ab} M_{ab}^2 - \frac{1}{4!} B_0 B^4 \sum_a \prod_{i=1} M_i^{\pm}. \tag{27}$$

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[2] The term proportional to C can be understood as a mass term for a scalar particle corresponding to an end of an open strings, see [6] for details.

Since the lattice is regular, the sum over loops can be thought of as a sum over plaquetes. We can define a plaquete operator $W_{a'}$ anchored at a point a' as

$$W_{a'} = M^{+}_{a'b} M^{-}_{bc} M^{+}_{cd} M^{-}_{da'}. \tag{28}$$

The points $a', \dots d$ are now fixed by a convention of labeling plaquettes given their base point so that there is no summation over repeated indices. With the help of this operator, the Hamiltonian can be written as follows

$$H_{lowT} \sim D \sum_{ab} M^2_{ab} - \frac{1}{3!} B_0 B^4 \sum_{a'} (W_{a'} + h.c.) \tag{29}$$

where $h.c.$ stands for the Hermitian conjugate of $W_{a'}$, i.e. a loop operator with M^+ and M^- interchanged on each link. The sum in the second term is over plaquettes.

It turns out that this Hamiltonian correspond to $U(1)$ gauge theory in axial $A_0 = 0$ gauge. In fact, the Kogut-Susskind Hamiltonian [9] for a gauge field on a cubic lattice in three spatial dimensions is

$$H_{KS} = \frac{g^2}{2a} {\sum_{ab}}' M^2_{ab} - \frac{2}{ag^2} \sum_{a'} (W_{a'} + h.c.). \tag{30}$$

The sum in the first term is shown primed because it is only over nearest neighbors connection in the lattice. The variables a and g denote the lattice spacing and the coupling constant, respectively. Comparing coefficients of our model and the gauge theory Hamiltonian gives the identifications

$$g^2 \sim \sqrt{\frac{4! D}{B_0 B^4}}, \qquad \frac{1}{a^2} \sim \frac{1}{3!} D B_0 B^4. \tag{31}$$

Recall that string-net condensation, and thus the possibility of emergent $U(1)$ theory, only happens when certain conditions such as (25) are satisfied. Inserting (25) into the expression (31) for g^2 above gives

$$g^2 = (2\gamma)^{-1/2}. \tag{32}$$

Recall that $\gamma > 1$, so that the coupling g^2 of the emergent gauge field is weak. Furthermore, if we set the inverse lattice spacing on the order of the Planck mass, $a^{-1} \sim m_P$, then it follows from (31) and (25) that

$$B_0 B^4 \sim m_P g^{-2}, \qquad D \sim g^2 m_P. \tag{33}$$

To be sure, the correspondence between H_{lowT} and H_{KS} is not exact because in the pure gauge theory, the edges connecting lattice sites can carry arbitrary representations of $U(1)$ whereas the m labels in our model can only take three values $-1, 0$, and 1. Nonetheless, the string condensed phase of H_{lowT} should exhibit features of $U(1)$ theory, such as the presence of photon-like excitations, even in this crude approximation [10]. The correspondence could be improved by allowing a wider range of m values on each edge. This is not in principle an obstacle for our model, but we keep the present setup for simplicity.

V. QUANTUM GRAPHITY AS A MODEL OF THE VERY EARLY UNIVERSE

Based on the quantum graphity model we have described in this paper we can propose the following scenario for the early history of the universe.

At early times, when $T \gg V$, the graph is in a very disordered state and the average valence of each node is large. The diameter of the graph, or the distance measured in "on" links between any two points, is approximately $\log(N)$ in this phase so that the degrees of freedom on a typical graph quickly come into thermal contact. Thus, the whole system may be assumed to come into thermal equilibrium.

As the system cools and the temperature drops, however, one or more phase transitions may occur in which the j degrees of freedom will become frozen. How the system cools depends on the relations between different coupling constants. We assume that the first transition that occurs is one in which the valence of each node becomes frozen to v_0. Thus as the temperature cools below V the spins arrange themselves into regular patterns that can be interpreted as extended space. We thus have the emergence of classical geometry as well as standard gauge theory and matter fields.

Even if this is a simplified model of the emergence of space, it suggests insights for physical cosmology. The horizon problem is the statement that distant parts of the universe appear to be in thermal equilibrium despite universe's evolution suggesting that these parts could not have interacted during the course of the universe's estimated lifetime. This is deemed to be a puzzle because its most straight-forward resolution by positing special initial conditions lacks physical justification. Our model provides such a justification because it suggests that the spins were part of a thermal ensemble before the temperature fell sufficiently for the system to enter a phase of classical geometry. Thus the model shows the horizon problem may be avoided if geometry is emergent. In this sense, the model also provides an explicit example of a broader idea that a distinction between micro-locality (locality between fundamental degrees of freedom) and macro-locality (locality between emergent degrees of freedom) may be important for understanding quantum gravity and the physics of the very early universe [7].

The model also allows us to discuss an important issue for quantum gravity models, which is the role of diffeomorphism invariance. When a notion of geometry emerges in the low energy limit, each state of the system corresponds to a geometry, which is a description of metric and fields modulo diffeomorphisms. This is because there is no role for diffeomorphism transformations in the original model, because it makes no reference to geometry.

To see this in detail we consider the case of the honeycomb lattice. We notice it can be naturally embedded in a flat two-dimensional space. Thus, it is possible to associate to it a flat Minkowski metric g which assigns lengths to the spins and positions to the points. (In fact, we made this assignment in the previous analysis by defining a spacing variable a.) Consider now a diffeomorphism φ acting on g. The invariance of the spin model under such deformations is illustrated in a concrete example of a φ with compact support in Fig. 3, where we see that φ moves the links and vertices but preserves the connectivity of the various elements. Moreover, the relation between excitations is unchanged, and since observables can only be defined in terms of relative excitations and connectivity of the graph without reference to the embedding, the dynamics and predictions made before or after the diffeomorphism are the same.

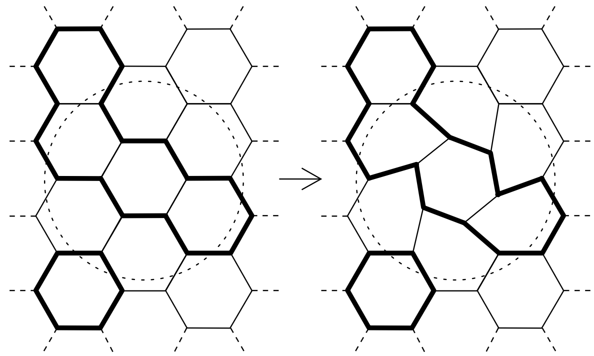

FIG. 3: Diffeomorphism invariance on the honeycomb lattice. The map is trivial outside the dotted region and is a twirl or rotation inside that region. Bold lines indicate strings.

VI. CONCLUSIONS

In this paper we have presented a class of models in which there is no notion of geometry fundamentally, but which allow geometry to emerge as an approximate description of a low temperature phase. We have conjectured that the low temperature phase is characterized by the formation of large, regular lattices. While we motivated this conjecture more work would need to be done to test it.

It is interesting to compare our model to the usual theory of gravity and matter defined by the Einstein-Hilbert and matter Lagrangian,

$$S = \frac{1}{4\pi G}\int d^4x\sqrt{-g}\,(R+2\Lambda) + \int d^4x\sqrt{-g}\,\mathcal{L}_M. \tag{34}$$

This formalism admits several limits which enable to study subparts of the full theory. On one hand, when the matter component is ignored by setting $\mathcal{L}_M = 0$, the leftover Einstein-Hilbert action is well defined and describes the dynamics of the gravitational field by itself. Indeed, there are many interesting and physically relevant solutions to the pure Einstein-Hilbert theory when $\mathcal{L}_M = 0$. On the other hand, the no-gravity limit $G \to 0$ is also well defined and describes a quantum field theory on a fixed background $g_{\mu\nu}$.

This kind of splitting does not occur in our model because the matter degrees of freedom play an essential role in organizing the links into a regular lattice structure. We find that if we try to construct a discrete space in a background independent manner, it is helpful to assign both j and m variables to the spins and write a Hamiltonian that acts on them individually. It turns out that the dynamics of the m variables that organizes the "on" links automatically gives us a $U(1)$ gauge theory. Alternatively, if we try to formulate a string condensation model of a $U(1)$ gauge theory so that it does not depend on a fixed lattice, we find that it is merely necessary to introduce one more state ($|0,\,0\rangle$) in the Hilbert space of each spin and to fix the valence of the nodes. Viewed in this way, the Hamiltonian of the quantum model is rather economical.

Apart from constructing a geometry, we have not attempted to reproduce features of gravitational physics such as gravitons in our model. This could be attempted using again string condensation techniques, following [11, 12], or perhaps using loop quantum gravity as a guide.

In the future we hope to use this model to study features of the conjectured geometrogenesis phase transition. As argued in [13] and [14] some aspects of the transition may be measurable. These include the transition temperature and a critical exponent that governs the speed of the transition and the proportion of non-local links left over after the transition.

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